

EVERY $K(n)$ -LOCAL SPECTRUM IS THE HOMOTOPY FIXED POINTS OF ITS MORAVA MODULE

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ABSTRACT. Let $n \geq 1$ and let p be any prime. Also, let E_n be the Lubin-Tate spectrum, G_n the extended Morava stabilizer group, and $K(n)$ the n th Morava K -theory spectrum. Then work of Devinatz and Hopkins and some results due to Behrens and the first author of this note, show that if X is a finite spectrum, then the localization $L_{K(n)}(X)$ is equivalent to the homotopy fixed point spectrum $(L_{K(n)}(E_n \wedge X))^{hG_n}$, which is formed with respect to the continuous action of G_n on $L_{K(n)}(E_n \wedge X)$. In this note, we show that this equivalence holds for *any* (S -cofibrant) spectrum X . Also, we show that for all such X , the strongly convergent Adams-type spectral sequence abutting to $\pi_*(L_{K(n)}(X))$ is isomorphic to the descent spectral sequence that abuts to $\pi_*((L_{K(n)}(E_n \wedge X))^{hG_n})$.

1. INTRODUCTION

In this note, we extend a result about the $K(n)$ -localization of finite spectra, which is due to a combination of work by Devinatz and Hopkins (in [4]) and Behrens and the first author of this note (in [1], [2]) (with most of the hard work being done by Devinatz and Hopkins), to all $K(n)$ -local spectra.

In more detail, let $n \geq 1$ and let p be a prime. Above and elsewhere, $K(n)$ denotes the n th Morava K -theory spectrum,

$$G_n = S_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

is the n th extended Morava stabilizer group, and E_n is the n th Lubin-Tate spectrum, with

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][[u^{\pm 1}]],$$

where $W(\mathbb{F}_{p^n})$ is the ring of Witt vectors with coefficients in the field \mathbb{F}_{p^n} , the degree of u is -2 , and the complete power series ring is in degree zero.

Given a spectrum X , we define

$$L_{K(n)}(E_n \wedge X),$$

the Bousfield localization of $E_n \wedge X$ with respect to $K(n)$, to be the *point-set level Morava module* of X . When $\pi_*(L_{K(n)}(E_n \wedge X))$ satisfies certain hypotheses, it is common for these stable homotopy groups to be referred to as the *Morava module* of X . However, in this note, since we never use the term “Morava module” in this algebraic sense, we will henceforth always refer to the point-set level Morava module of X as just its Morava module.

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By [5], G_n acts on E_n through maps of commutative S -algebras and, by regarding X as having trivial G_n -action and then giving its Morava module the diagonal G_n -action, $L_{K(n)}(E_n \wedge X)$ is a G_n -spectrum.

Since the time of [10] and the circulation of the results of [11], it has been believed by many experts in chromatic homotopy theory that it ought to be possible to realize the $K(n)$ -localization of any spectrum X as the G_n -homotopy fixed points of some E_n -module spectrum that is built out of E_n and X . However, to date, it has not been clear how to do this.

Tremendous progress towards such a result was made by [4], which showed that

$$(1.1) \quad E_n^{dhG_n} \simeq L_{K(n)}(S^0),$$

where $E_n^{dhG_n}$ is a commutative S -algebra that behaves like a G_n -homotopy fixed point spectrum (e.g., the associated $K(n)$ -local E_n -Adams spectral sequence looks like a descent spectral sequence, with E_2 -term equal to continuous cohomology).

Notice that equivalence (1.1) implies that whenever X is a finite spectrum,

$$E_n^{dhG_n} \wedge X \simeq L_{K(n)}(X).$$

Additionally, by [2, Theorem 1.3], the Morava module $L_{K(n)}(E_n \wedge X) \simeq E_n \wedge X$ (since X is finite) is a continuous G_n -spectrum, so that its G_n -homotopy fixed point spectrum $(L_{K(n)}(E_n \wedge X))^{hG_n}$ can be formed. Also, by [1, Theorem 8.2.1], there is an equivalence

$$E_n^{dhG_n} \simeq E_n^{hG_n}.$$

Taken together, the preceding conclusions imply that $E_n^{hG_n} \wedge X \simeq L_{K(n)}(X)$, and hence, since X is a finite spectrum,

$$(1.2) \quad (L_{K(n)}(E_n \wedge X))^{hG_n} \simeq L_{K(n)}(X),$$

by [2, Theorem 9.9]. Therefore, the $K(n)$ -localization of any finite spectrum can be realized as the G_n -homotopy fixed points of its Morava module.

Now we are ready to explain how our last conclusion is generalized in this note. First, we remark that, from this point onward, we always work in the stable model category of symmetric spectra of simplicial sets or in its homotopy category. Thus, a “spectrum” is a symmetric spectrum of simplicial sets and, when we work with, for example, commutative algebras, these objects are always to be understood as referring to commutative algebras in the setting of symmetric spectra.

Recall that [2, Theorem 1.3] shows that for any spectrum X , the Morava module $L_{K(n)}(E_n \wedge X)$ is a continuous G_n -spectrum, where, as before, X is regarded as having the trivial G_n -action and the Morava module has the diagonal G_n -action. Then, in this note, we generalize equivalence (1.2) in the following way.

Theorem 1.3. *If X is any S -cofibrant spectrum, then*

$$(L_{K(n)}(E_n \wedge X))^{hG_n} \simeq L_{K(n)}(X).$$

We quickly make a technical (but useful) comment about Theorem 1.3. By using cofibrant replacement in the S model structure on the category of symmetric spectra, given any spectrum Z , there is a weak equivalence $Z_c \rightarrow Z$ in the usual stable model category of symmetric spectra, with Z_c S -cofibrant (see [7, Section 5.3]). Thus, there is no loss of generality in Theorem 1.3 in requiring that X be S -cofibrant, so that the theorem can be thought of as being valid for an arbitrary spectrum X .

Theorem 1.3 shows that the $K(n)$ -localization of any (S -cofibrant) spectrum is the G_n -homotopy fixed points of its Morava module, answering the relatively old question of how to show that every $K(n)$ -local spectrum can be obtained from a homotopy fixed point construction involving E_n and G_n . The proof of Theorem 1.3 is given in Section 2.

We give Theorem 1.3 the desired (but generally unwieldy) “computational legs” with the following result.

Theorem 1.4. *If X is any S -cofibrant spectrum, then the strongly convergent $K(n)$ -local E_n -Adams spectral sequence abutting to $\pi_*(L_{K(n)}(X))$ is isomorphic to the descent spectral sequence that abuts to $\pi_*((L_{K(n)}(E_n \wedge X))^{hG_n})$, from the E_2 -terms onward.*

If X is a finite spectrum, then Theorem 1.4 reduces to [1, Theorem 8.2.5]. We refer the reader to [4, Appendix A] for an exposition of the aforementioned Adams-type spectral sequence. The descent spectral sequence that Theorem 1.4 refers to is defined in [1, Section 4.6]. The proof of Theorem 1.4 is given in Section 3.

Acknowledgements. The proof of Theorem 1.3 that appears in this note is a simplified version of an argument that relied more heavily on some results from [1]. Thus, the first author thanks Mark Behrens for various things that he learned from him during their collaboration on [1].

2. THE PROOF OF THEOREM 1.3

We begin this section by establishing some notation. We let

$$c: \widehat{S} \xrightarrow{\sim} L_{K(n)}(S^0)$$

be a cofibrant replacement of $L_{K(n)}(S^0)$ in the model category of commutative symmetric ring spectra (see [8, the discussion just before Theorem 19.6]); the map c is a weak equivalence in the stable model category of symmetric spectra. (We need the cofibrant commutative symmetric ring spectrum \widehat{S} because later we will regard it as the ground ring of a profinite Galois extension.) Also, for the remainder of this section, we write K in place of $K(n)$, so that our notation does not become too cumbersome. We will sometimes use the terminology of [9, Section 1], adapted to the K -local category, as in [4, Appendix A].

As in [1, Section 5.2], let $\mathcal{A}l_g$ be the model category of discrete commutative G_n - \widehat{S} -algebras: the objects of $\mathcal{A}l_g$ are discrete G_n -spectra that are also commutative \widehat{S} -algebras, and the morphisms are G_n -equivariant maps of commutative \widehat{S} -algebras. Let $(-)_f: \mathcal{A}l_g \rightarrow \mathcal{A}l_g$ be a fibrant replacement functor for the model category $\mathcal{A}l_g$, and let $N \triangleleft_o G_n$ denote an open normal subgroup of G_n . Also, recall from [4] that each E_n^{dhN} – the commutative \widehat{S} -algebra that is written as E_n^{hN} in [4] and behaves like the N -homotopy fixed point spectrum of E_n – is a G_n/N -spectrum that is K -local. Then, as in [2] and [1], let

$$F_n = \operatorname{colim}_{N \triangleleft_o G_n} (E_n^{dhN})_f;$$

by construction, F_n is a discrete G_n -spectrum and a commutative symmetric ring spectrum that is $E(n)$ -local. Here, $E(n)$ is the usual Johnson-Wilson spectrum, with

$$\pi_*(E(n)) = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}][v_n^{\pm 1}].$$

Let X be any S -cofibrant spectrum. By [2, Theorem 9.7],

$$(L_K(E_n \wedge X))^{hG_n} \simeq L_K((F_n \wedge X)^{hG_n}),$$

where $(F_n \wedge X)^{hG_n}$ is the G_n -homotopy fixed points of the discrete G_n -spectrum $F_n \wedge X$. Thus, to prove Theorem 1.3, we only have to show that

$$L_K((F_n \wedge X)^{hG_n}) \simeq L_K(X).$$

Let $L_K((E_n)^{\wedge(\bullet+1)})$ be the usual cosimplicial spectrum that is built from the unit map $S^0 \rightarrow E_n$ and the multiplication $E_n \wedge E_n \rightarrow E_n$. Here, for each $k \geq 0$,

$$L_K((E_n)^{\wedge(k+1)}) = L_K(\underbrace{E_n \wedge E_n \wedge \cdots \wedge E_n}_{k+1})$$

and there is the associated augmented resolution

$$(2.1) \quad * \rightarrow \widehat{S} \rightarrow E_n \rightarrow L_K(E_n \wedge E_n) \rightarrow L_K(E_n \wedge E_n \wedge E_n) \rightarrow \cdots,$$

which is the canonical K -local E_n -resolution of \widehat{S} .

Given \widehat{S} -modules M and N , we let $M \wedge_{\widehat{S}} N$ denote their smash product in the category of \widehat{S} -modules. Then resolution (2.1) can be identified with the resolution

$$(2.2) \quad * \rightarrow \widehat{S} \rightarrow E_n \rightarrow L_K(E_n \wedge_{\widehat{S}} E_n) \rightarrow L_K(E_n \wedge_{\widehat{S}} E_n \wedge_{\widehat{S}} E_n) \rightarrow \cdots,$$

since, for each $k \geq 0$,

$$L_K((E_n)^{\wedge(k+1)}) \simeq L_K((E_n)^{\wedge_{\widehat{S}}(k+1)}).$$

Since $E_n \simeq L_K(F_n)$ (this equivalence is due to [4], but the reader might find the proof of it in [2, Theorem 6.3] useful), resolution (2.2) can be identified with the resolution

$$(2.3) \quad * \rightarrow \widehat{S} \rightarrow L_K(F_n) \rightarrow L_K(F_n \wedge_{\widehat{S}} F_n) \rightarrow L_K(F_n \wedge_{\widehat{S}} F_n \wedge_{\widehat{S}} F_n) \rightarrow \cdots.$$

For the next step in our proof, we make a few recollections. We define the cosimplicial spectrum $\text{Map}^c(G_n^\bullet, F_n)$ as in [1, Section 3.2]: if K is a profinite group, then

$$\text{Map}^c(K, F_n) = \text{colim}_{U \triangleleft_o K} \text{Map}(K/U, F_n) \cong \text{colim}_{U \triangleleft_o K} \prod_{K/U} F_n;$$

$\text{Map}^c(G_n, -)$ is a coaugmented comonad on the category of spectra; and, via the comonadic cobar construction, $\text{Map}^c(G_n^\bullet, F_n)$ is the associated cosimplicial spectrum, which, in each codegree k , satisfies the isomorphism

$$(\text{Map}^c(G_n^\bullet, F_n))^k \cong \text{Map}^c(G_n^k, F_n).$$

We recall that by [12, Theorem 5.4.4] and [1, pg. 5034], the work of [4] and [3] implies that F_n is a K -local profinite G_n -Galois extension of \widehat{S} of finite vcd (in the sense of [1]). Thus, as in the proof of [1, Proposition 6.3.1], there is an equivalence

$$L_K((F_n)^{\wedge(\bullet+1)}) \simeq L_K(\text{Map}^c(G_n^\bullet, F_n))$$

of cosimplicial spectra. (Here, if a map $X^\bullet \rightarrow Y^\bullet$ of cosimplicial spectra is a weak equivalence in each codegree, then we regard it as a weak equivalence of cosimplicial spectra.) Therefore, resolution (2.3) is equivalent to the resolution

$$(2.4) \quad * \rightarrow \widehat{S} \rightarrow L_K(F_n) \rightarrow L_K(\text{Map}^c(G_n, F_n)) \rightarrow L_K(\text{Map}^c(G_n^2, F_n)) \rightarrow \cdots,$$

and hence, resolution (2.4) can be regarded as the canonical K -local E_n -resolution of \widehat{S} .

Below, to save space, we sometimes use the notation $Y \hat{\wedge} Z$ to denote $L_K(Y \wedge Z)$, where Y and Z are arbitrary spectra, and, for the same reason, we occasionally write $(-)_K$ in place of $L_K(-)$. By smashing resolution (2.4) with X and then localizing with respect to K , it follows from [4, Remark A.9] that

$$* \rightarrow X_K \rightarrow (F_n)_K \hat{\wedge} X \rightarrow \mathrm{Map}^c(G_n, F_n)_K \hat{\wedge} X \rightarrow \mathrm{Map}^c(G_n^2, F_n)_K \hat{\wedge} X \rightarrow \cdots$$

is a K -local E_n -resolution of $L_K(X)$. Thus, the equivalent resolution

$$(2.5) \quad * \rightarrow X_K \rightarrow F_n \hat{\wedge} X \rightarrow \mathrm{Map}^c(G_n, F_n) \hat{\wedge} X \rightarrow \mathrm{Map}^c(G_n^2, F_n) \hat{\wedge} X \rightarrow \cdots$$

is a K -local E_n -resolution of $L_K(X)$, so that, since (2.5) is the resolution associated to the canonical map

$$\Phi: L_K(X) \rightarrow \mathrm{holim}_{\Delta} L_K(\mathrm{Map}^c(G_n^{\bullet}, F_n) \wedge X),$$

Φ is a weak equivalence, by [4, Corollary A.8].

Since F_n is $E(n)$ -local and $L_{E(n)}(-)$ is a smashing localization, $F_n \wedge X$ is also $E(n)$ -local. Then the proof of Theorem 1.3 is finished by noting that

$$\begin{aligned} L_K((F_n \wedge X)^{hG_n}) &\simeq L_K(\mathrm{holim}_{\Delta} \mathrm{Map}^c(G_n^{\bullet}, F_n \wedge X)) \\ &\simeq \mathrm{holim}_{\Delta} L_K(\mathrm{Map}^c(G_n^{\bullet}, F_n \wedge X)) \\ &\simeq \mathrm{holim}_{\Delta} L_K(\mathrm{Map}^c(G_n^{\bullet}, F_n) \wedge X) \\ &\simeq L_K(X), \end{aligned}$$

where the first equivalence follows immediately from [1, Theorem 3.2.1]; the second equivalence follows from the observation that $\mathrm{Map}^c(G_n^{\bullet}, F_n \wedge X)$ is a diagram of $E(n)$ -local spectra (see, for example, [1, Corollary 6.1.3; Lemma 6.1.4, (3)]); and the third equivalence uses the fact that for each $k \geq 1$, there are natural equivalences

$$\begin{aligned} \mathrm{Map}^c(G_n^k, F_n \wedge X) &\simeq \mathrm{colim}_{U \triangleleft_o G_n^k} \bigvee_{G_n^k/U} (F_n \wedge X) \\ &\cong \mathrm{colim}_{U \triangleleft_o G_n^k} ((\bigvee_{G_n^k/U} F_n) \wedge X) \\ &\simeq \mathrm{colim}_{U \triangleleft_o G_n^k} ((\prod_{G_n^k/U} F_n) \wedge X) \\ &\cong \mathrm{Map}^c(G_n^k, F_n) \wedge X, \end{aligned}$$

where the third equivalence (just above, between the two colimits) uses the fact that, since X is S -cofibrant, the functor $(-) \wedge X$ preserves weak equivalences, by [7, Corollary 5.3.10].

3. THE PROOF OF THEOREM 1.4

Let X be any S -cofibrant spectrum. In Section 2, we showed that resolution (2.5) is a $K(n)$ -local E_n -resolution of $L_{K(n)}(X)$. Thus, by [4, discussion preceding Proposition A.5], there is a map ϕ from the strongly convergent $K(n)$ -local E_n -Adams spectral sequence for $\pi_*(L_{K(n)}(X))$, which we denote by ${}^{\mathcal{A}}E_r^{*,*}(X)$, to the homotopy spectral sequence for

$$\pi_*(\mathrm{holim}_{\Delta} L_{K(n)}(\mathrm{Map}^c(G_n^{\bullet}, F_n) \wedge X)),$$

which we denote by ${}^I E_r^{*,*}(X)$. Furthermore, by [4, Proposition A.5], the map ϕ of spectral sequences is an isomorphism, from the E_2 -terms onward.

Notice that there are equivalences

$$\begin{aligned} L_{K(n)}(\mathrm{Map}^c(G_n^\bullet, F_n) \wedge X) &\simeq L_{K(n)}(\mathrm{Map}^c(G_n^\bullet, F_n \wedge X)) \\ &\simeq \mathrm{holim}_i \mathrm{Map}^c(G_n^\bullet, F_n \wedge X \wedge M_i) \end{aligned}$$

of cosimplicial spectra, where the first equivalence was obtained at the end of Section 2 and the second equivalence is an application of the fact that for any $E(n)$ -local spectrum Z , $L_{K(n)}(Z) \simeq \mathrm{holim}_i (Z \wedge M_i)$, where

$$M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_i \leftarrow \cdots$$

is a suitable tower of generalized Moore spectra (see [6, Section 2]).

The above equivalences of cosimplicial spectra imply that ${}^I E_r^{*,*}(X)$ is isomorphic to the homotopy spectral sequence for

$$\pi_* \left(\mathrm{holim}_{\Delta} \mathrm{holim}_i \mathrm{Map}^c(G_n^\bullet, F_n \wedge X \wedge M_i) \right),$$

which we denote by ${}^{II} E_r^{*,*}(X)$. By [1, Section 4.6], ${}^{II} E_r^{*,*}(X)$ is exactly the descent spectral sequence for $\pi_*((L_{K(n)}(E_n \wedge X))^{hG_n})$, so that the chain of isomorphisms

$${}^A E_r^{*,*}(X) \cong {}^I E_r^{*,*}(X) \cong {}^{II} E_r^{*,*}(X)$$

completes the proof of Theorem 1.4.

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